1201. We need to simplify

$$f^2(x) = \frac{1}{1 + \frac{1}{1+x}}.$$

Using the standard technique, we multiply top and bottom of the large fraction by the denominator of the small fraction, which gives

$$f^2(x) = \frac{1+x}{1+x+1} \equiv \frac{x+1}{x+2}.$$

1202. The line is $y = a + \frac{b-a}{h}x$. So, the area is

$$\begin{aligned} A &= \int_0^h a + \frac{b-a}{h}x \, dx \\ &\equiv \left[ax + \frac{b-a}{2h}x^2\right]_0^h \\ &\equiv ah + \frac{b-a}{2h}h^2 \\ &\equiv \tfrac{1}{2}(2a + b - a)h \\ &\equiv \tfrac{1}{2}(a+b)h, \text{ as required.} \end{aligned}$$

1203. Each numerator has roots $x = \pm 1$. So, the overall equation will have exactly these roots, unless the denominator also does.

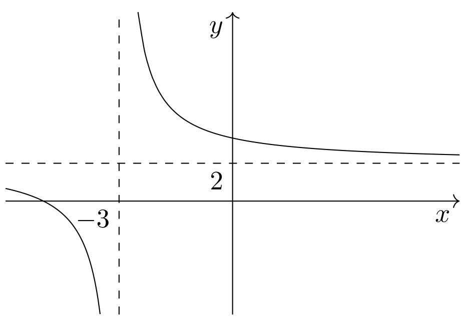
(a) True, as the denominator is never zero.
(b) False, as the denominator is zero at $x = 0, -1$.
(c) True, as the denominator is zero at $x = \pm\sqrt{2}$.

1204. There are infinitely many we could construct. For example, $\pi/1000 = 0.003141...$ is less than 0.01, and is irrational. Hence, the sum $3.6 + \pi/1000$ satisfies the conditions of the question.

1205. (a) We write $2x + 7 \equiv 2(x+3) + 1$. Hence, the graph is

$$y = 2 + \frac{1}{x+3}.$$

(b) The curve is a transformed reciprocal graph $y = 1/x$. Replacing x by $x + 3$ translates the curve by $-3\mathbf{i}$; adding two to the output of the fraction translates the graph by $2\mathbf{j}$. Hence, the asymptotes are at $x = -3$ and $y = 2$:



1206. No two lines are parallel, so every pair of lines crosses exactly once. There are ${}^{100}C_2$ such pairs. And, since no three lines are concurrent, the points of intersection are all distinct. Hence, the total number is ${}^{100}C_2 = 4950$ intersections.

1207. (a) For $y = mx + c$, $\dfrac{dy}{dx} = m$.

(b) Substituting into the DE,

$$\begin{aligned} m &\equiv 2x - 3(mx + c) \\ \implies 3mx + m &\equiv 2x - 3c. \end{aligned}$$

(c) Since the above is an identity, we can equate coefficients of x^1 and x^0. This gives

$$\begin{aligned} x^1 &: 3m = 2 \\ x^0 &: m = -3c. \end{aligned}$$

Hence, $m = \frac{2}{3}$ and $c = -\frac{2}{9}$.

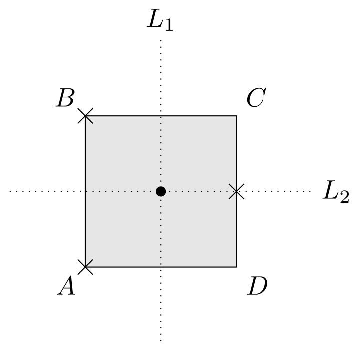
$$\begin{aligned} y &= \tfrac{2}{3}x - \tfrac{2}{9} \\ \implies 9y &= 6x - 2 \\ \implies 9y - 6x + 2 &= 0. \end{aligned}$$

So, $9y - 6x + 2 = 0$ is the only linear solution.

1208. Rearranging and factorising,

$$\begin{aligned} & x^5 - 8x^2 = 0 \\ \implies & x^2(x^3 - 8) = 0 \\ \implies & x^2 = 0 \text{ or } x^3 = 8 \\ \implies & x = 0, 2. \end{aligned}$$

1209. (a) All forces are vertical. So, we represent them as points on the base. The crosses represent (upwards/out of the page) reaction forces, the dot represents the (downwards/into the page) weight:



Since L_2 is a line of symmetry, the forces R_A and R_B must be equal.

(b) Taking moments about L_1,

$$2R_A = R_{CD}.$$

Then NII perpendicular to $ABCD$ gives

$$2R_A + R_{CD} = mg.$$

Solving these, $R_A = R_B = \frac{1}{4}mg$, $R_{CD} = \frac{1}{2}mg$.

1210. The integrand is the y value of the semicircle

$$y = \sqrt{1 - x^2}.$$

Hence, the value of the integral is the area of a unit semicircle, which is $\pi/2$.

1211. (a) Since $x = k$ is a line parallel to the y axis, it must intersect a curve such as this one, which is defined for all real values of x.

(b) The equation for intersections is

$$\begin{aligned} y^2 + k + 1 &= k \\ \Longrightarrow y^2 + 1 &= 0. \end{aligned}$$

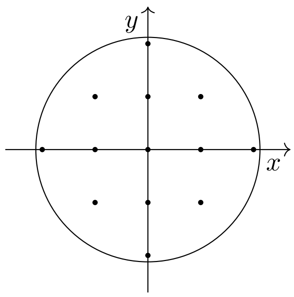
Since $y^2 + 1$ is always positive, this has no real roots. Hence, there are no intersections.

1212. The relationship between a, the side length, and b, the diagonal length, is $b = \sqrt{2}a$. Differentiating this with respect to time tells us that the rates of change are related as

$$\frac{db}{dt} = \sqrt{2} \times \frac{da}{dt}.$$

We are told that $\dfrac{da}{dt} = 2$, so $\dfrac{db}{dt} = 2\sqrt{2}$ cm/s.

1213. The inequality is the interior of a circle centred at the origin. The relevant points then form a square, with vertices at $(0, \pm 2)$ and $(\pm 2, 0)$. There are 13 points in the square.



1214. (a) Differentiating, $f'(x) = 3x^2 - 10x - 2$. So, the N-R iteration is

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n^2 - 2x_n - 24}{3x_n^2 - 10x_n - 2}.$$

(b) With $x_0 = 0$, the N-R iteration gives $x_n \to 6$. Substituting this into the original function,

$$f(6) = 6^3 - 5 \cdot 6^2 - 2 \cdot 6 - 24 = 0.$$

This verifies that $x = 6$ is a root of $f(x)$.

(c) The factor theorem states that, if $x = \alpha$ is a root of a polynomial, then $(x - \alpha)$ is a factor.

(d) Taking out the linear factor,

$$f(x) = (x - 6)(x^2 + x + 4).$$

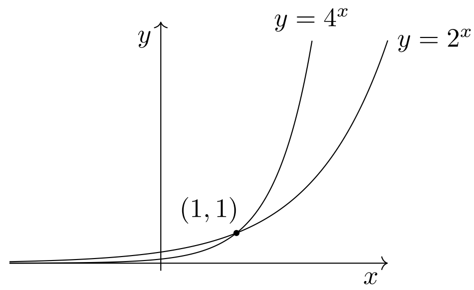
The discriminant of the quadratic factor is $\Delta = -15 < 0$. Hence, the quadratic has no real roots. Thus, over the reals $\mathbb{R}$, no further factorisation is possible.

1215. The exponential graphs $y = 2^x$ and $y = 4^x$ both pass through $(0, 1)$. The latter is

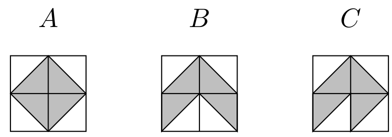
$$y = 4^x \equiv \left(2^2\right)^x \equiv 2^{2x}.$$

So, $y = 4^x$ is $y = 2^x$ after a stretch, by scale factor $\frac{1}{2}$, in the x direction.

To get $y = 2^{x-1}$ and $y = 4^{x-1}$, both graphs are then translated by **i**, since x has been replaced by $x - 1$ in each. This gives



1216. (a) There are three cases:



There is one outcome of type A. Type B may be rotated, giving 4 outcomes. Type C may be rotated and reflected, giving 8 outcomes. $1 + 4 + 8 = 13$, as required.

(b) Each small square has 4 possible orientations, so there are $4^4 = 256$ possible configurations. Of these, 13 are successful. So, the required probability is $\frac{13}{256}$.

1217. Enacting the operation "differentiate with respect to x", as encoded in the differential operator $\frac{d}{dx}(\,)$,

$$\begin{aligned} \frac{d}{dx}(x^2 + 2y) &= 10 \\ \Longrightarrow 2x + 2\frac{dy}{dx} &= 10 \\ \Longrightarrow \frac{dy}{dx} &= 5 - x. \end{aligned}$$

1218. (a) The acceptance region (a, b) is bounded at both end; the critical region $(-\infty, a] \cup [b, \infty)$ consists of two regions (tails). So, the test is two-tailed.

(b) The acceptance region has 99% probability, so the critical region has 1% probability. This is a significance level of 1%.

1219. We know that $u_n = ar^{n-1}$, for some $a, r \in \mathbb{R}$. To prove that w_n is also geometric, we need to prove that the ratio of consecutive terms is constant. So, consider

$$\begin{aligned}&\frac{w_{n+1}}{w_n}\\ &= \frac{pu_{n+1} + qu_{n+2}}{pu_n + qu_{n+1}}\\ &= \frac{par^n + qar^{n+1}}{par^{n-1} + qar^n}\\ &\equiv \frac{pr + qr^2}{p + qr}.\end{aligned}$$

Since p, q, r are all constant, this is constant.

1220. The statement is false. Surds provide the natural counterexamples: $\sqrt{2} \times \sqrt{2} = 2$.

1221. (a) This is not true. If f and g are the same linear function, then E has solution set $\mathbb{R}$.

(b) This is true. If $s \in S$, then $f(s) = g(s) = 0$, so s must also be in the solution set of E.

(c) This is not true. If f and g are the same linear function, then E has solution set $\mathbb{R}$.

1222. Writing the fractions with negative indices,

$$\begin{aligned}&\int_1^2 x^{-2} + 2x^{-3}\,dx\\ &= \Big[-x^{-1} - x^{-2}\Big]_1^2\\ &= \left(-2^{-1} - 2^{-2}\right) - \left(-1^{-1} - 1^{-2}\right)\\ &= \left(-\tfrac{3}{4}\right) - (-2)\\ &= \tfrac{5}{4}, \text{ as required.}\end{aligned}$$

1223. Yes, this is entirely possible. A "reaction" force is any contact force which is perpendicular to the surfaces in contact. Two floating astronauts, even if they are weightless, can exert reaction forces on one another by pushing.

1224. Half of the shaded area is a segment subtending an angle of 120° at the centre of the associated semicircle. The associated sector has area $\pi/3$. The associated triangle has area $\frac{1}{2}\sin 120° = \sqrt{3}/4$. Therefore, the area of the shaded region is

$$\tfrac{2}{3}\pi - \tfrac{\sqrt{3}}{2}.$$

The area of the unshaded region is that of two semicircles minus that of the shaded region:

$$\begin{aligned}A &= \pi - 2\left(\tfrac{2}{3}\pi - \tfrac{\sqrt{3}}{2}\right)\\ &= \sqrt{3} - \tfrac{1}{3}\pi, \text{ as required.}\end{aligned}$$

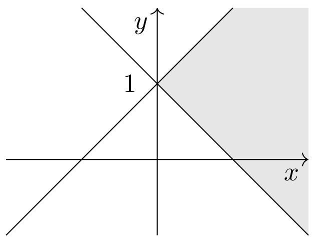
1225. The odd numbers are a sequence with first term 1 and common difference 2. Using the standard formula, the sum is

$$\begin{aligned}S_{1000} &= \tfrac{1000}{2}\big(2\cdot 1 + (1000-1)2\big)\\ &= 1000000.\end{aligned}$$

1226. The functions f, g, h have 0, 1, 2 roots respectively. Since the solution sets are distinct, we can add the numbers of roots of each factor:

(a) $0 + 1 = 1$ root.
(b) $0 + 1 + 2 = 3$ roots.
(c) $1 + 0 = 1$ root, since $h(x)^{-1}$ can never be zero.

1227. The boundary lines are $y = x + 1$ and $y = 1 - x$. These intersect at $(0, 1)$. So, the region is



1228. The formula for variance is

$$s^2 = \frac{\sum x^2 - n\bar{x}^2}{n}.$$

This gives

$$\begin{aligned}0.9011 &= \frac{227 - n\cdot 1.17^2}{n}\\ \Longrightarrow 0.9011n &= 227 - 1.17^2 n\\ \Longrightarrow n &= \frac{227}{0.9011 + 1.17^2}\\ &= 100.\end{aligned}$$

1229. The total score is $a+b$, which is out of $A+B$. The scale factor required to take the number of total marks available to 100, therefore, is

$$\frac{100}{A+B}.$$

So, the formula is

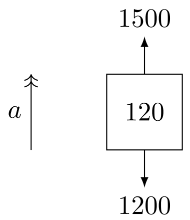
$$X = \frac{100(a+b)}{A+B}.$$

1230. Assume, for a contradiction, that a pair of distinct circles has three distinct intersections P, Q, R. Let L_1 be the perpendicular bisector of PQ and L_2 be the perpendicular bisector of PR. The lines L_1 and L_2 intersect at C, which is therefore the centre of both circles.

The circles intersect at P, Q, R, so they must also have the same radius. Hence, they are the same circle, i.e. not distinct. This is a contradiction.

Hence, a pair of distinct circles cannot have more than two intersections. QED.

1231. (a) The force diagram for the first stage is



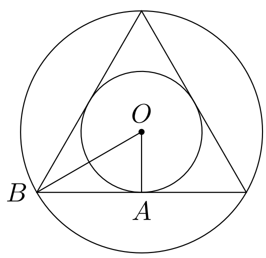
$1500 - 1200 = 120a \implies a = 2.5 \text{ ms}^{-2}$. Then, $s = ut + \frac{1}{2}at^2$ gives $500 = \frac{1}{2} \times 2.5 \times t^2$. Taking the positive square root, $t = 20$ seconds.

(b) When the fuel runs out, the vertical speed is $u = 20 \times 2.5 = 50 \text{ ms}^{-1}$, and the height is 50 ms^{-1}. So, for hitting the ground,

s	-50
u	50
v	v
a	-10
t	not relevant

$v^2 = u^2 + 2as$ gives $v^2 = 50^2 + 2 \cdot -10 \cdot -50$. The speed at which the rocket hits the ground is $v = 59.160... = 59.2 \text{ ms}^{-1}$ (3sf).

1232. Labelling O, A, B, the scenario is



Triangle OAB has angles $(30°, 60°, 90°)$. This gives $|OA| = \frac{1}{2}|OB|$. Since the ratio of radii is $1 : 2$, the ratio of areas is therefore $1 : 4$, as required.

1233. (a) A_n has ordinal formula $A_n = pn^2 + qn + r$, for some $p, q, r \in \mathbb{R}$ with $p \neq 0$. Hence, $(A_n)^2$ is quartic, which is neither arithmetic nor quadratic.

(b) B_n has ordinal formula $B_n = pn + q$, for some $p, q \in \mathbb{R}$, with $p > 0$. Hence, $(B_n)^2$ is quadratic.

1234. There are two possibilities, BR and RB. Each has probability $\frac{6\times4}{10\times9}$. Hence, the total probability is

$$\mathbb{P}(\text{different colours}) = 2 \times \frac{6 \times 4}{10 \times 9} = \frac{8}{15}.$$

1235. The transformations between the two curves are reflection in the x axis followed by translation in the y direction by c. Neither of these affects the x coordinates of points on the curve. Hence, the SPs will remain at the same x values. □

— Alternative Method —

Since the derivative of an additive constant is zero, the gradient formulae are negatives of one another: $\frac{dy}{dx} = \pm f'(x)$. Equated to zero, these produce the same equation for SPs. Hence, the x values of the SPs are the same. □

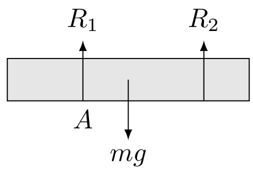
1236. This is true iff the relevant graph has the y axis as a line of symmetry, because, e.g. substituting $-30°$ into (b), we require $\cos 30° = \cos -30°$. Only cosine has such even symmetry:

(a) No.

(b) Yes.

(c) No.

1237. The force diagram is as follows:



$$\updownarrow: \quad R_1 + R_2 - mg = 0,$$
$$\overset{\curvearrowleft}{A}: \quad mg - 3R_2 = 0$$

Solving these, $R_2 = \frac{1}{3}mg$ and $R_1 = \frac{2}{3}mg$.

1238. Factorising the top and bottom,

$$\begin{aligned} &\lim_{x\to2} \frac{2x^2 + 6x - 20}{3x^2 - 7x + 2} \\ &= \lim_{x\to2} \frac{(2x-4)(x+5)}{(3x-1)(x-2)} \\ &= \lim_{x\to2} \frac{2(x+5)}{3x-1}. \end{aligned}$$

At this point, having cancelled factors of $(x - 2)$, we can take the limit. This gives

$$\lim_{x\to2} \frac{2(x+5)}{3x-1} = \frac{14}{5}.$$

1239. Factorising explicitly,

$$2x^3 - 7x^2 - 53x + 28 \equiv (2x^2 + 7x - 4)(x - 7).$$

— Alternative Method —

The quadratic factorises as

$$2x^2 + 7x - 4 = (2x - 1)(x + 4).$$

Testing $x = \frac{1}{2}$ and $x = -4$,

$$\begin{aligned} 2\left(\tfrac{1}{2}\right)^3 - 7\left(\tfrac{1}{2}\right)^2 - 53\left(\tfrac{1}{2}\right) + 28 = 0, \\ 2(-4)^3 - 7(-4)^2 - 53(-4) + 28 = 0. \end{aligned}$$

By the factor theorem, $(2x - 1)$ and $(x + 4)$ are factors of the cubic, so $(2x + 7x - 4)$ is a factor of the cubic.

1240. (a) $a \in [1, 10)$.

(b) Using log rules, we can express $\log_p z$ as

$$\log_p \left(a \times 10^b\right) \equiv \log_p a + b \log_p 10.$$

This is of the form $mb + c$, with $m = \log_p 10$ and $c = \log_p a$, so $\log_p z$ is linear in b.

1241. Setting $\Delta = b^2 - 4ac = 121$, we get $49 - 24k = 121$. Hence, $k = -3$.

1242. Suppose the shape is $ABCD$, with AC taken to be the line of symmetry of the kite. Since $ABCD$ is a parallelogram, $|AB| = |CD|$ and $|AD| = |BC|$. Furthermore, since $ABCD$ is a kite, $|AB| = |AD|$. Hence,

$$|AB| = |BC| = |CD| = |DA|.$$

So, the shape is a rhombus. QED.

1243. (a) Strip height.

(b) Strip area.

(c) The total area of the strips.

(d) The area beneath the graph.

— Nota Bene —

If one takes thinner and thinner rectangles (letting $\delta x \to 0$), while taking more of them, then the total area of the rectangles approaches the area under the curve. So, in the limit, the identity holds.

1244. Setting $x = 2$ gives $0 = 20 + 4b$. Hence, $b = -5$. Factorising the RHS, $a = 3$.

1245. The vertical height of the ladder, in metres, is

$$2.4 \sin 60° = \tfrac{\sqrt{3}}{2} \times 2.4 = \tfrac{6\sqrt{3}}{5}.$$

We need to show that this is greater than 2, hence that its square is greater than 4. Squaring the height gives

$$\left(\tfrac{6\sqrt{3}}{5}\right)^2 = \tfrac{36 \times 3}{25} = \tfrac{108}{25} = 4\tfrac{8}{25} > 4.$$

1246. Expanding the brackets and differentiating,

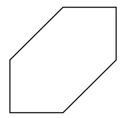
$$\begin{aligned} & y = x^{\frac{3}{2}} - x^{\frac{1}{2}} \\ \implies & \frac{dy}{dx} = \tfrac{3}{2}x^{\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}}. \end{aligned}$$

So, the limit in question is

$$\lim_{x \to 0} \frac{dy}{dx} = \lim_{x \to 0} \left(\tfrac{3}{2}x^{\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}}\right).$$

The second term is a reciprocal: $\frac{1}{2}x^{-\frac{1}{2}} \equiv \frac{1}{2\sqrt{x}}$. Such a term grows without bound as $x \to 0$.

1247. If P holds, then every pair of opposite sides must be parallel and the same length, which gives Q. So, $P \implies Q$. The other two implications don't hold. A counterexample is $\mathbf{a} = \mathbf{i}, \mathbf{b} = \mathbf{i} + \mathbf{j}, \mathbf{c} = \mathbf{j}$:



1248. Using two log rules,

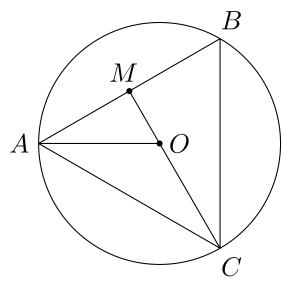
$$\begin{aligned} & 2 \log_x 2 = 1 - 2 \log_x 3 \\ \implies & \log_x 4 = 1 - \log_x 9 \\ \implies & \log_x 4 + \log_x 9 = 1 \\ \implies & \log_x 36 = 1 \\ \implies & x = 36. \end{aligned}$$

1249. (a) Completing the square, we have

$$F[y = (x-1)^2 + 6] = (1, 6).$$

(b) $y = \mathrm{g}(x)$ is a monic quadratic with vertex at $(3, 0)$. Hence, $\mathrm{g}(x) = (x-3)^2$.

1250. If A is a vertex, M the midpoint of side AB, and O the centre of the triangle ABC, then triangle AMO has angles $(30°, 60°, 90°)$.



In triangle AMO, length $|AM| = 1$, so

$$\cos 30° = \tfrac{\sqrt{3}}{2} = \frac{1}{|AO|}$$

Therefore $|AO| = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.
So, the equilateral triangle will fit exactly inside a circle of radius $\frac{2\sqrt{3}}{3}$, as required.

1251. This is the sum to infinity S_∞ of a GP with first term $a = \frac{3}{10}$ and common ratio $r = \frac{1}{10}$. Hence, using the standard formula $S_\infty = \frac{a}{1-r}$,

$$S_\infty = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \tfrac{1}{3}.$$

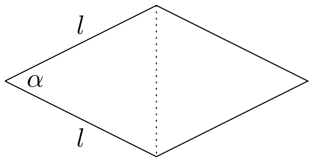
— Alternative Method —

Writing the series longhand,

$$\sum_{k=1}^{\infty} \frac{3}{10^k} = 0.3 + 0.03 + 0.003 + 0.0003 + ...$$

This is $0.\dot{3}$, which is $\frac{1}{3}$.

1252. The rhombus looks as follows:



Using the formula $A = \frac{1}{2}ab\sin C$, the area to the left of the dotted line is $\frac{1}{2}l^2 \sin\alpha$. Hence, the full area is given by $A = l^2 \sin\alpha$. □

1253. Translating into algebra, we want to find x such that $1/x = \ln x$. So, we set $\text{f}(x) = 1/x - \ln x$. The Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{\text{f}(x_n)}{\text{f}'(x_n)}.$$

For this f, the N-R iteration is

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - \ln x_n}{-\frac{1}{x_n^2} - \frac{1}{x_n}}.$$

This simplifies to

$$x_{n+1} = x_n + \frac{\frac{1}{x_n} - \ln x_n}{\frac{1}{x_n^2} + \frac{1}{x_n}}.$$

Running the iteration with $x_1 = 1$, we get $x_2 = 1.5$ and $x_n \to 1.76322228...$. So, the root is $x = 1.76$ to 3sf. We can verify this with error bounds:

$$\begin{aligned} \text{f}(1.755) &= 0.0733... > 0 \\ \text{f}(1.765) &= -0.00157... < 0. \end{aligned}$$

1254. The given equation could not generate the graph. The equation $(x+1)(x-1)^2 = 0$ has a single root at $x = -1$ and a double root at $x = +1$, but the graph shows the opposite: a single root (crossing) at $x = +1$ and a double root (tangency) at $x = -1$.

1255. The implication goes forwards. By squaring both sides, we can see that $-x^3 = 3 \implies x^6 = 9$. But the reverse implication doesn't hold, as 9 has two square roots: the counterexample is $x = \sqrt[3]{3}$, for which $x^6 = 9$, but $-x^3 = -3$.

1256. This is false. Factorising,

$$\begin{aligned} y &= (x^2 - 3)(x^2 + 2) \\ &\equiv \left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)(x^2 + 2). \end{aligned}$$

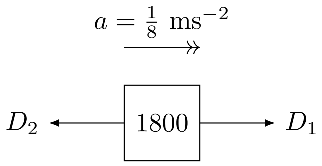
This has two single roots at $x = \pm\sqrt{3}$. For the x axis to be tangent to $y = \text{f}(x)$, we would require a repeated root.

1257. (a) $4 \times ① - ②$ gives the required result.

(b) $③ - ②$ gives $2a - c = -3$.

(c) Solving these two equations simultaneously, we get $a = 1, c = 5$. Substituting this back in gives $b = 1$.

1258. The acceleration is given by $s = ut + \frac{1}{2}at^2$, which is $1 = \frac{1}{2}a \cdot 16$. So, $a = \frac{1}{8}$ ms^{-2}. Modelling the entire scrum, then, considering only horizontal forces,



NII gives $D_1 - D_2 = 1800 \times \frac{1}{8} = 225$ N.

1259. The quadratic ordinal formula is

$$u_n = an^2 + bn + c.$$

The first differences are 8 and 10, so the second difference is $2 = 2a$. Hence, $a = 1$. The first term gives $7 = 1 + b + c$, the second $15 = 4 + 2b + c$. Solving simultaneously, $b = 5$ and $c = 1$. Hence, the hundredth term is

$$u_{100} = 100^2 + 5 \cdot 100 + 1 = 10501.$$

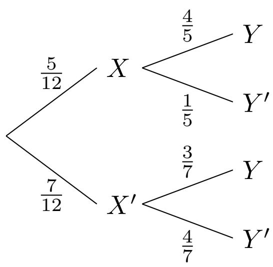
1260. We begin with $\mathbb{P}(X) = \frac{5}{12}$ and $\mathbb{P}(X') = \frac{7}{12}$. Then, in the circle representing X, we have

$$\begin{aligned} \mathbb{P}(Y \mid X) &= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{12}} = \frac{4}{5} \\ \mathbb{P}(Y' \mid X) &= \frac{\frac{1}{12}}{\frac{1}{3} + \frac{1}{12}} = \frac{1}{5}. \end{aligned}$$

Equivalent calculations in X' give

$$\begin{aligned} \mathbb{P}(Y \mid X') &= \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{3}} = \frac{3}{7}, \\ \mathbb{P}(Y' \mid X') &= \frac{\frac{1}{3}}{\frac{1}{4} + \frac{1}{3}} = \frac{4}{7}. \end{aligned}$$

So, the tree diagram is



1261. Since the units digit of n^2 depends only on the units digit of n, we need only check the squares of $0, 1, ..., 9$. These are $0, 1, 4, 16, 25, 36, 49, 64, 81$, none of which ends in $2, 3, 7$ or 8. So, no square number ends in $2, 3, 7$ or 8. □

1262. Multiplying up and gathering like terms,

$$\begin{aligned} y &= \frac{a\sqrt{x}+b}{c\sqrt{x}+d} \\ \implies cy\sqrt{x}+dy &= a\sqrt{x}+b \\ \implies cy\sqrt{x}-a\sqrt{x} &= b-dy. \end{aligned}$$

Taking out a factor of $\sqrt{x}$,

$$\begin{aligned} \sqrt{x}(cy-a) &= b-dy \\ \implies \sqrt{x} &= \frac{b-dy}{cy-a} \\ \implies x &= \left(\frac{b-dy}{cy-a}\right)^2 \end{aligned}$$

1263. The LHS is

$$\begin{aligned} &\big(a^2+b^2\big)\big(c^2+d^2\big) \\ \equiv\ & a^2c^2+a^2d^2+b^2c^2+b^2d^2. \end{aligned}$$

The RHS is

$$\begin{aligned} &\big(ac-bd\big)^2+\big(ad+bc\big)^2 \\ \equiv\ & a^2c^2-2abcd+b^2d^2+a^2d^2+2abcd+b^2c^2 \\ \equiv\ & a^2c^2+a^2d^2+b^2c^2+b^2d^2. \end{aligned}$$

LHS $\equiv$ RHS, proving the identity.

1264. Completing the square for both x and y gives

$$(x+a)^2+(y+b)^2=a^2+b^2.$$

So, the radius of the circle is $\sqrt{a^2+b^2}$. Inscribing a square, the right-angled triangle of two radii and one edge has sides in the ratio $1:1:\sqrt{2}$. So, the edges of the square have length $\sqrt{2}\sqrt{a^2+b^2}$. Hence, the area of the square is $2(a^2+b^2)$, as required.

1265. The implication is $x\in A \implies x\in A\cup B$. If x is an element of A, then it must be in A's union with any set. The reverse is not true, however. For a counterexample, consider any $x\in B\setminus A$.

1266. $4^{2x+3}\equiv\big(2^2\big)^{2x+3}\equiv 2^{4x+6}\equiv 2^{4x}\cdot 2^6\equiv 64\big(2^x\big)^4.$

1267. Since "no scores are odd" is the same as "all scores are even", the two events have equal probability.

1268. Multiplying top and bottom of the large fractions by $\sqrt{c}$ gives

$$\frac{\sqrt{c}+1}{\sqrt{c}-1}+\frac{\sqrt{c}-1}{\sqrt{c}+1}.$$

Over a common denominator, this is

$$\frac{(\sqrt{c}+1)^2+(\sqrt{c}-1)^2}{(\sqrt{c}+1)(\sqrt{c}-1)}\equiv\frac{2c^2+2}{c-1}.$$

1269. Differentiating the proposed solution curve,

$$\frac{dy}{dx}=2x.$$

Substituting into the DE,

$$\begin{aligned} 2x &\equiv \sqrt{4(x^2+c)-20} \\ \implies 4x^2 &\equiv 4x^2+4c-20. \end{aligned}$$

So, we require $4c-20=0$, giving $c=5$.

1270. (a) Vertically, $s=-1.6$, $u=0$, $a=-g$. So,

$$\begin{aligned} -1.6 &= \tfrac{1}{2}\cdot -gt^2 \\ \implies t^2 &= \tfrac{16}{49} \\ \therefore\quad t &= \tfrac{4}{7}. \end{aligned}$$

The time of flight is $t=\frac{4}{7}$ seconds.

(b) At $t=\frac{4}{7}$, the components of the velocity are $v_x=5$, $v_y=-g\times\frac{4}{7}=-5.6$. Hence, the angle below the horizontal is

$$\arctan\tfrac{5.6}{5}=48.239...=48.2^\circ\ (1\text{dp}).$$

1271. (a) Each rectangle has area 96 cm^2. Therefore, calling the area of the unshaded rectangle A, $2\times 96-2A=100$. Solving, we get $A=46$ cm^2.

(b) There isn't enough information to find the perimeter. The unshaded rectangle, with an area of 100 cm^2, could have different dimensions, and hence perimeters.

1272. Using the binomial expansion,

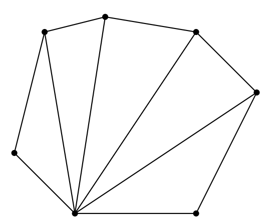
$$\big(\sqrt{x}+1\big)^4\equiv x^2+4x\sqrt{x}+6x+4\sqrt{x}+1.$$

Subtracting this from $4\sqrt{x}+4x\sqrt{x}$, the odd terms cancel. The original equation simplifies to

$$\begin{aligned} x^2+6x+2 &= 0 \\ \implies x &= \frac{-6\pm\sqrt{36-4\cdot 2}}{2} \\ &= -3\pm\sqrt{7}. \end{aligned}$$

However, both of these roots are negative, and are not in the domain of the square root function. Hence, the equation has no roots.

1273. By drawing diagonals from a vertex to every other vertex, any convex n-gon can be split up into $n-2$ triangles. The interior angles of these triangles all lie at the vertices of the n-gon.



Hence, the sum of the interior angles of the $n-2$ triangles is equal to the sum of the interior angles of the n-gon. Each triangle has π radians, giving $(n-2)\pi$ radians overall. □

1274. This algebraic fraction simplifies to a polynomial iff $(x-3)$ is a factor of the numerator. We can check this using the factor theorem. Substituting $x=3$ into the numerator gives

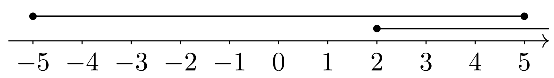
$$16\cdot 3^4+5\cdot 3^2-6\cdot 3+1=1324.$$

Since this is non-zero, $(x-3)$ is not a factor. Hence, the algebraic fraction cannot be simplified to a polynomial.

1275. Using 3D Pythagoras, the distance is

$$d=\sqrt{\left(\tfrac{l}{2}\right)^2+\left(\tfrac{l}{2}\right)^2+\left(\tfrac{l}{2}\right)^2}=\tfrac{\sqrt{3}}{2}l.$$

1276. The first set $\{z\in\mathbb{R}:|z|\leq 5\}$ can be written as $[-5,5]$. So, the two sets are



Removing the lower set from the upper set,

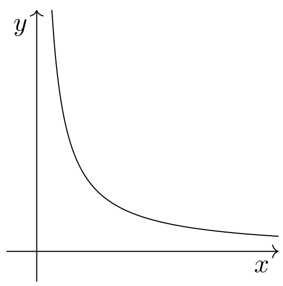
$$\{z\in\mathbb{R}:|z|\leq 5\}\setminus[2,\infty)=[-5,2).$$

1277. Setting the differences equal to each other,

$$\begin{aligned}&2p+1-p=2p^2-3-(2p+1)\\ \Longrightarrow\ &2p^2-3p-5=0\\ \Longrightarrow\ &(2p-5)(p+1)=0\\ \Longrightarrow\ &p=\tfrac{5}{2},-1.\end{aligned}$$

1278. (a) Using log rules, we can write $\log_2(xy)=1$. Then, exponentiating both sides, base 2, we get $xy=2$, which is inverse proportion.

(b) The graph is a standard reciprocal $xy=2$, apart from the fact that x and y, as inputs to a logarithm function, must be positive. Hence, the graph is in the positive quadrant:



1279. Using the binomial expansion, we have

$$(x\pm h)^4=x^4\pm 4x^3h+6x^2h^2\pm 4xh^3+1.$$

Hence, we can simplify to

$$\begin{aligned}&\lim_{h\to 0}\frac{8x^3h+8xh^3}{2h}\\ =&\lim_{h\to 0}4x^3+4xh^2\\ =&\ 4x^3,\ \text{as required.}\end{aligned}$$

— Nota Bene —

This is differentiation from first principles. In this version, points are taken either side of x, at $x-h$ and $x+h$. These are $2h$ apart. Both points then tend to x as $h\to 0$, so the gradient of the chord tends to the gradient of the tangent at x.

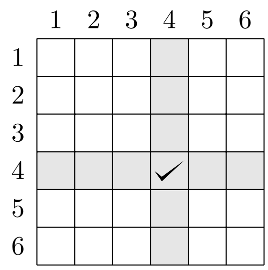
1280. (a) Yes. The graph is a positive parabola.

(b) Yes. The graph has intercepts at $x=\pm k$, so

$$y=a(x+k)(x-k)\equiv ax^2-ak^2.$$

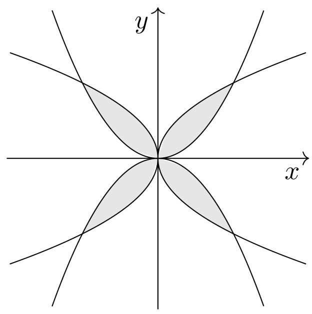
(c) Yes. The y intercept is negative.

1281. The information given restricts the possibility space to the shaded outcomes, of which only one is successful.



So, $\mathbb{P}(\text{both fours}\mid\text{at least one four})=\frac{1}{11}$.

1282. (a) The design is



(b) The area of the leaf in the positive quadrant is given by the following integral:

$$\begin{aligned}&\int_0^1\sqrt{x}-x^2\,dx\\ =&\left[\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{3}x^3\right]_0^1\\ =&\left(\tfrac{2}{3}-\tfrac{1}{3}\right)-(0)\\ =&\ \tfrac{1}{3}.\end{aligned}$$

Hence, the total shaded area is $\frac{4}{3}$.

1283. Assume, for a contradiction, that

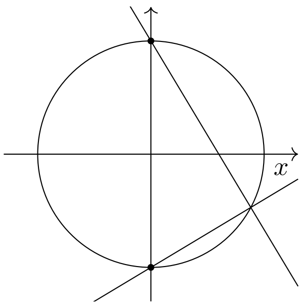
$$(2x^2 - x + 1)(ax^2 + bx + c) \equiv 2x^4 - 3x^3 + 2.$$

Equating coefficients,

$$\begin{aligned} &x^4 : a = 1, \\ &x^3 : 2b - a = -3, \text{ so } b = -1, \\ &x^2 : 2c - b + a = 0, \text{ so } c = -1. \end{aligned}$$

But then the constant term is -1 on the LHS and $+2$ on the RHS. Hence, such a factorisation is not possible.

1284. Two of the vertices are on the line $x = 0$; they have coordinates $(0, \pm 1)$. This side is a diameter of the circle. So, by the angle in a semicircle theorem, the other two sides are perpendicular.



The second side has gradient $\frac{3}{5}$ and y intercept -1. So, the third side has gradient $-\frac{5}{3}$ and y intercept 1; its equation is $y = 1 - \frac{5}{3}x$, as required.

1285. Using the binomial distribution

$$\mathbb{P}(X = 40) = {}^{100}\mathrm{C}_{40} \times 0.5^{100} = 0.0108439$$

Using the normal distribution

$$\mathbb{P}(39.5 < Y < 40.5) = 0.0108521$$

The values are approximately equal.

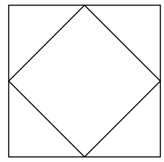
— Nota Bene —

This is the basis for the normal approximation to the binomial distribution: if $X \sim B(n, p)$, for large n and for p not too close to either 0 or 1, then X can be approximated by $Y \sim N(np, npq)$.

1286. The factor theorem tells us that, if a polynomial $\mathrm{f}(x)$ has a factor of $x - b$, then $\mathrm{f}(b) = 0$. Hence $b^2 + 4b + a = 0$. This is a quadratic in b. Using the quadratic formula,

$$b = \frac{-4 \pm \sqrt{16 - 4a}}{2} = -2 \pm \sqrt{4 - a}.$$

1287. The largest such inscribed square has its vertices at the midpoints of the sides of the unit square:



By Pythagoras, the length scale factor between the two squares is $1 : \sqrt{2}$, so the area scale factor is $1 : 2$. Hence, the area is $1/2$. The lower bound (not attainable) is 0, so the set is $(0, 1/2]$.

— Nota Bene —

It would be reasonable to give the answer $\left(0, 1/2\right)$ here, taking the question to mean that S must be *strictly* inside U.

1288. The coordinates of the endpoints, at $t = \pm 1$, are

$$(p \pm \cos\theta, q \pm \sin\theta).$$

So, the vector between them is $2\cos\theta\mathbf{i} + 2\sin\theta\mathbf{j}$. Finding its length l, we get $l^2 = 4(\cos^2\theta + \sin^2\theta)$. The identity $\sin^2\theta + \cos^2\theta \equiv 1$ gives $l = 2$.

— Alternative Method —

For $\Delta t = 1$, the displacement is $\cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. This has unit length. Hence, over the interval $[-1, 1]$, which has t-width 2, the distance is 2.

1289. Differentiating with respect to u,

$$\frac{dx}{du} = \cos u.$$

Reciprocating and squaring both sides gives

$$\left(\frac{du}{dx}\right)^2 = \sec^2 u.$$

The Pythagorean trig identity $\tan^2 u + 1 \equiv \sec^2 u$ gives the required result.

1290. (a) We integrate both sides, combining constants of integration into one on the RHS. This gives $\mathrm{f}(x) = \mathrm{g}(x) + c$. So, $\mathrm{f}(x) - \mathrm{g}(x) = c$.

(b) A log rule has been misused. The true rule is

$$\log_a \frac{x}{y} \equiv \log_a x - \log_a y,$$

using which the logarithm of a fraction may be expressed as a difference of logarithms. The student has instead expressed the logarithm of a difference $\log_a(\mathrm{f}(x) - \mathrm{g}(x))$ as a fraction of logarithms $\log_a \mathrm{f}(x) / \log_a \mathrm{g}(x)$. The result proposed is not, in fact, true.

1291. Vertex A can be placed anywhere, without loss of generality. The probability, then, that chord AB crosses the diameter is the probability that B is on the opposite side to A. There remain 13 vertices to be chosen from, of which 7 are on the opposite side, so the probability is $\frac{7}{13}$.

1292. The mean interior angle in a quadrilateral is $\frac{\pi}{2}$. Since these interior angles form an AP, we know that the mean of the smallest and largest angles is therefore $\frac{\pi}{2}$. So, their sum is π radians.

1293. (a) The acceleration is given by

$$a = \tfrac{1}{2}t^{-\frac{1}{2}} - 1.$$

So, maximum velocity is attained where

$$\tfrac{1}{2}t^{-\frac{1}{2}} - 1 = 0.$$

This is at $t = 1/4$ seconds. Substituting, we get $v_{\max} = 1/4 \text{ ms}^{-1}$.

(b) Integrating v with limits of $t = 0$ and $t = \frac{16}{9}$ gives the displacement over that interval:

$$\begin{aligned} &\int_0^{\frac{16}{9}} \sqrt{t} - t \, dt \\ &= \left[\tfrac{2}{3}t^{\frac{3}{2}} - \tfrac{1}{2}t^2\right]_0^{\frac{16}{9}} \\ &= \tfrac{2}{3} \cdot \tfrac{64}{27} - \tfrac{1}{2} \cdot \tfrac{256}{81} \\ &= 0. \end{aligned}$$

Hence, the object returns to O at $t = \frac{16}{9}$.

1294. Completing the square, the circle is

$$\left(x + \tfrac{1}{2}\right)^2 + \left(y - 2\right)^2 = \tfrac{9}{4}.$$

So, the centre of the circle is $C : \left(-\frac{1}{2}, 2\right)$. The squared distances of the given points from C are

$$\tfrac{1}{2}^2 + 4^2 = \tfrac{65}{4} \text{ and } \tfrac{7}{2}^2 + 2^2 = \tfrac{65}{4}.$$

So, the points are equidistant from the circle.

1295. Since $\overrightarrow{OA}$ is a unit vector, A must lie on the unit circle $x^2 + y^2 = 1$. We can solve simultaneously. Substituting for y,

$$\begin{aligned} &x^2 + (1 - 2x)^2 = 1 \\ \Longrightarrow\ &5x^2 - 4x = 0 \\ \Longrightarrow\ &x = 0, 0.8. \end{aligned}$$

Possible coordinates are $(0, 1)$ and $(0.8, -0.6)$.

1296. In each case, it is the parity (oddness/evenness) of the index k in $(x - 1)^k$ that determines whether a sign change takes place.

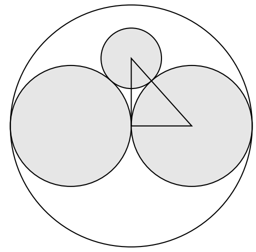
(a) Yes, $k = 1$, which is odd.
(b) Yes, $k = -1$, which is odd.
(c) No, $k = -2$, which is even.

1297. Exponentiating both sides,

$$\begin{aligned} &e^{\ln y} = e^{3\ln x + 4} \\ \Longrightarrow\ &y = e^{\ln x^3 + 4} \\ \Longrightarrow\ &y = e^4 x^3. \end{aligned}$$

Since e^4 is a constant, $y \propto x^3$.

1298. To contain the two large circles, the surrounding circle must be centred at their point of tangency:



The result is shown visually above. To prove it geometrically, consider the vertical height of the triangle, which is

$$\sqrt{15^2 - 10^2} = \sqrt{125} < 12.$$

Hence, the highest point is at a vertical height of $\sqrt{125} + 5 < 17$ above the centre, so will fit inside a circle of radius 20.

1299. Equating differences gives $b - a = (a + b) - b$ and $(a + b) - b = 8 - (a + b)$. These simplify to $b = 2a$, $2a = 8 - b$. Solving simultaneously, $a = 2$ and $b = 4$.

1300. The sum of three consecutive cubes is given by

$$n^3 + (n + 1)^3 + (n + 2)^3,$$

where $n \in \mathbb{Z}$. The binomial expansions are

$$\begin{aligned} (n + 1)^3 &\equiv n^3 + 3n^2 + 3n + 1, \\ (n + 2)^3 &\equiv n^3 + 6n^2 + 12n + 8. \end{aligned}$$

So, the sum is

$$\begin{aligned} &3n^3 + 9n^2 + 15n + 9 \\ \equiv\ &3(n^3 + 3n^2 + 5n + 3). \end{aligned}$$

This is divisible by 3, as required.

——— End of 13th Hundred ———